Exercise Sheet 2: Cofibrations

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Contents

1 Instructions

Please read and attempt to understand all of this exercise sheet. *You are asked to submit two exercises from each of the three sections.* There are six total in §2, three total in §3, and three total in §4 to choose from.

2 Cofibrations I

For these exercises we will be working in Top, the category of unbased spaces. In particular all maps and homotopies will be assumed to be free.

Definition 1 *A map $j : A \to X$ between spaces A, X is said to have the* ***homotopy extension property*** *with respect to a space Y if for each pair of map $f : X \to Y$ and a homotopy $F : A \times I \to Y$ starting at $F_0 = fj$, there exists a homotopy $\widetilde{F} : X \times I \to Y$ such that*

1) $\widetilde{F}_0 = f$

2) $\widetilde{F}_t j = F_t$.

We say that j is a ***cofibration*** *if it has the homotopy extension property with respect to all spaces.* $\square$

The situation is as follows:

$$\begin{tikzcd} A \arrow[r, hook, "in_0"] \arrow[d, "j"'] & A \times I \arrow[d, "j\times 1"'] \arrow[ddr, bend left, "F"] & \\ X \arrow[r, "in_0"] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "\widetilde{F}"] & \\ & & Y. \end{tikzcd} \tag{2.1}$$

The map j is a cofibration if, whenever the solid part of the diagram commutes, then the dotted extension may be filled in. By taking adjoints we can equivalently consider this as a lifting problem

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "j"'] & Y^I \arrow[d, "e_0"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "\widetilde{F}"] & Y. \end{tikzcd} \tag{2.2}$$

Why introduce cofibrations? Here is some motivation. It is a classical problem in the case that $j : A \hookrightarrow X$ is a subspace inclusion to ask when a given map $f : A \to Y$ may be extended over all of X

$$\begin{tikzcd} A \arrow[d, hook, "j"'] \arrow[dr, "f"] & \\ X \arrow[r, dashed] & Y. \end{tikzcd} \tag{2.3}$$

If j is a cofibration, then the problem becomes a question about the homotopy class of f. For in this case, if f is homotopic to some map which does extend, then f also can be extended. Thus cofibrations were introduced as a means to convert the topological problem into one approachable by homotopy-theoretic methods. It will turn out that cofibrations have much further reaching application.

So, having persuaded ourselves that cofibrations might be useful, let's look for some examples.

Example 2.1

1) For any space X, the inclusion $\emptyset \hookrightarrow X$ is a cofibration.

2) A homeomorphism is a cofibration.

3) If $f : A \to X$ and $g : B \to Y$ are cofibrations, then $f \sqcup g : A \sqcup B \to X \sqcup Y$ is a cofibration.

Sadly while these three examples have been obvious, they have been less than interesting. To construct some more exciting examples we will need some tools. Return to the square (2.1). The condition for j to be a cofibration is similar to that for the square to be a pushout, only weaker in that no uniqueness is required of the map $\widetilde{F}$. In any case, the shape of the diagram suggests a way to proceed.

Definition 2 *The* ***mapping cylinder*** *of a map $j : A \to X$ is the space M_j defined by the pushout*

$$\begin{tikzcd} A \arrow[r, "in_0"] \arrow[d, "j"'] & A \times I \arrow[d] \\ X \arrow[r] & M_j . \arrow[ul, phantom, "\lrcorner" very near start] \end{tikzcd} \tag{2.4}$$

In particular we will understand

$$M_j = \frac{X \sqcup A \times I}{[j(a) \sim (a, 0)]}. \qquad \square \tag{2.5}$$

Now, using the diagram

$$\begin{tikzcd} A \arrow[r, "in_0"] \arrow[d, "j"'] & A \times I \arrow[d] \arrow[ddr, bend left, "j \times 1"] & \\ X \arrow[r] \arrow[drr, bend right, "in_0"'] & M_j \arrow[dr, dashed, "t"] & \\ & & X \times I \end{tikzcd} \tag{2.6}$$

we define a map $t : M_j \to X \times I$.

Proposition 2.1 *The following statements are equivalent for a given map $j : A \to X$.*

1) j is a cofibration.

2) j has the homotopy extension property with respect to the mapping cylinder M_j.

3) The map $t : M_j \to X \times I$ has a left inverse $r : X \times I \to M_j$[1].

Exercise 2.1 Prove Proposition 2.1. $\square$

Exercise 2.2 Using Proposition 2.1 show that:

1) The inclusion $0 \hookrightarrow I$ is a cofibration.

2) The inclusion $S^n \hookrightarrow D^{n+1}$ is a cofibration

3) For any space X, the inclusion $j_X : X \hookrightarrow \widetilde{C}X$ into its cone is a cofibration. $\square$

You have just demonstrated that several important maps are cofibrations. It is no accident they are all are inclusions.

Proposition 2.2 *A cofibration $j : A \to X$ is an embedding. If X is Hausdorff, then it is a closed embedding.*

We'd like to prove this. There is a direct approach, but our method will make use of a special type of limit.

[1]i.e. $rt = id_{M_j}$

Definition 3 *An* ***equaliser*** *for a parallel pair of maps $f, g : X \to Y$ consists of a space $E = E(f, g)$ together with a map $u : E \to X$ such that*

1) $fu = gu$.

2) For any space Z and any map $h : Z \to X$ with $fh = gh$, there exists a unique map $h' : Z \to E$ satisfying $uh' = h$.

$$\begin{tikzcd} E \arrow[r, "u"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ & Z \arrow[u, "h"'] \arrow[ul, dotted, "h'"] & \end{tikzcd} \tag{2.7}$$

□

Note that this is a categorical construction, and the definition makes sense in any category. Clearly when they exist equalisers must be unique up to isomorphism.

Exercise 2.3 Show that equalisers exist in Top. More specifically, with reference to (2.7), show that the subspace

$$E = \{x \in X \mid f(x) = g(x)\} \subseteq X \tag{2.8}$$

satisfies the universal property when u is the inclusion. Conclude that the equaliser (2.7) is an embedding, and is a closed embedding when both X, Y are Hausdorff[2] □

Exercise 2.4 Now prove Proposition 2.3: let $f, g : X \to X \times I$ be the maps

$$f(x) = (x, 1), \qquad g(x) = t(r(x, 1)) \tag{2.9}$$

where t, r are as in (2.6), and show that

$$A \xrightarrow{j} X \overset{f}{\underset{g}{\rightrightarrows}} X \times I \tag{2.10}$$

is an equaliser diagram. □

Cofibrations which are closed embeddings tend to have improved properties over those which are not. We call them **closed cofibrations**.

Warning Now that you know a cofibration is an embedding, when reading Proposition 2.1 you may be inclined to replace the mapping cylinder M_j with the subspace $X \times 0 \cup A \times I \subseteq X \times I$. There is no harm in doing this when $j : A \hookrightarrow X$ is a closed inclusion or a cofibration, but be careful about doing so in general. Let us explain.

If $j : A \xrightarrow{\subseteq} X$ is a subspace inclusion, then $t : M_j \to X \times I$ maps M_j bijectively onto the subspace $X \times 0 \cup A \times I$. If A is closed in X, then t is a homeomorphism onto its image. Similarly, if j is a cofibration then the presence of the retraction $t : X \times I \to M_j$ also guarantees that t is a homeomorphism onto its image. On the other hand, if j is neither closed nor a cofibration, then the pushout topology on M_j may not coincide with the product topology on $X \times 0 \cup A \times I$.

[2]The converse is also true. If $A \subseteq X$ is any subspace, then there is an equaliser diagram of the form $A \xrightarrow{\subseteq} X \rightrightarrows \{0, 1\}$ (can you define the maps in it?). This shows that the embeddings in Top are exactly those maps which equalise some pair of arrows.

Example 2.2 If j is the inclusion $j : (0,1] \subseteq [0,1]$, then the topology on its mapping cylinder is strictly finer than the product topology. In $([0,1] \times 0 \cup (0,1] \times I) \subseteq I \times I$ (product topology) the sequence $z_n = (\frac{1}{n}, \frac{1}{n})$, $n \in \mathbb{N}$, converges to $(0,0)$. On the other hand, in the mapping cylinder M_j (quotient topology), the sequence z_n does not converge to $(0,0)$, since this point has neighbourhoods which do not meet any point of the diagonal in $(0,1] \times I$. $\square$

Nevertheless, it is possible to show the following.

Theorem 2.3 *A subspace inclusion $j : A \hookrightarrow X$ is a cofibration if and only if $X \times 0 \cup A \times I$ is a retract of $X \times I$.* $\blacksquare$

The forwards implication of this follows from Proposition 2.1, as does the backwards implication in the case that A is closed in X. As our comments above suggest, it is only the remaining case which is non-trivial. The full proof is exceeding technical and we will not ask you to attempt it. For details we refer you to Strøm [5] Lemma 3, or [2] Anhang 1, where Strøm's proof is recreated.

Many people choose to work exclusively with Hausdorff spaces, and many people assume that all cofibrations are closed embeddings. Here are some examples to convince yourself that these things are not the same.

Exercise 2.5 Prove that:

1) Not every closed embedding is a cofibration. You may consider the subspace $\{0, \frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ and the inclusion $\{0\} \hookrightarrow \{0, \frac{1}{n} \mid n \in \mathbb{N}\}$.

2) Not every cofibration is closed. You may consider the Sierpinski space $\mathbb{S} = \{u, c\}$ where $\{u\}$ is open and $\{c\}$ is closed, and the inclusion $\{u\} \hookrightarrow \mathbb{S}$. $\square$

Finally we'll end this section with another characterisation theorem for cofibrations. This should be compared to Theorem 2.3, which gives simple geometric critera for a map to be a cofibration. The following replaces this appealing geometric picture with technical details. As these things tend to go, the proof is more difficult, but the result often proves to be a more powerful tool.

Theorem 2.4 (Strøm) *A subspace inclusion $j : A \hookrightarrow X$ is a cofibration if and only if there exists a map $\varphi : X \to I$, and a homotopy $H : X \times I \to X$, satisfying*

1) $A \subseteq \varphi^{-1}(0)$

2) $H_0 = id_X$

3) $H_t|_A = id_A$ *for all* $t \in I$.

4) $H(x,t) \in A$ *whenever* $t > \varphi(x)$.

We call the pair (φ, H) as in the theorem a **Strøm structure**. Your task now is to prove Strøm's cofibration characterisation theorem.

Exercise 2.6 We'll start with the forwards direction. Assume that j is a cofibration and choose a retraction $r : X \times I \to X \times 0 \cup A \times I$, which exists by Theorem 2.3. Show that the functions

1) $\varphi(x) = \sup_{t \in I} |t - pr_2(r(x,t))|$

2) $H = pr_1 \circ r$

define a Strøm structure for j[3].

Now lets do the backwards implication. Fix a Strøm structure for j and conclude the theorem by showing that

$$r(x,t) = \begin{cases} (H_t(x), 0) & t \leq \varphi(x) \\ (H_t(x), t - \varphi(x)) & t \geq \varphi(x)) \end{cases} \tag{2.11}$$

defines a retraction $r : X \times I \to (X \times 0) \cup (A \times I)$. $\square$

Here is a simple observation in the special case that A is closed in X. Firstly, if (φ, H) is a Strøm structure on $j : A \hookrightarrow X$, and A is closed, then we can assume that $A = \varphi^{-1}(0)$. For suppose that $x \in X$ and $\varphi(x) = 0$. Then $x = H(x,0) = \lim_{n \to \infty} H(x, 1/n)$. Since each term of the sequence $\{H(x,1/n)\}_{n \in \mathbb{N}}$ lies in the closed set A, so does the limit x.

Corollary 2.5 *A closed cofibration is the inclusion of the zero-set of a real-valued function.* $\blacksquare$

The second observation we would like to make is in the case that $\varphi < 1$ throughout X (we do not need to assume that A is closed for this). If this holds, then the Strøm structure exhibits A as a strong deformation retract of X. This has a partial converse: if $A \subseteq X$ is both a cofibration and a deformation retract, then it is always possible to find a Strøm structure (φ, H) with $\varphi < 1$ throughout X. Note that the word cofibration cannot be omited from this statement. Not every (strong) deformation retract is a cofibration.

Example 2.3 Let Ω be an uncountable indexing set and form the product $\prod_\Omega I$. The inclusion $\{0\} \hookrightarrow \prod_\Omega I$ is a strong deformation retract. However, it is not a cofibration. Reason: it is not a zero-set. You may consult [2] for a proof of this fact. $\square$

On pg. 114 of his book [3] Hatcher defines the notion of a *good pair*. Our example here shows that not every good pair comes from a cofibration. Similarly not every cofibration gives rise to a good pair, although those that are encountered in practice certainly do. In either case, it is the presence of a suitably nice neighbourhood which is key. If you revisit Example 1.1 of Lecture 3, where we studied the pointed comb space, you can see what problems the lack of one causes.

Example 2.4 Let X be a manifold (possibly with boundary) and $A \subseteq X$ a closed subset. A neighbourhood $N \subseteq X$ of A is said to be a **mapping cylinder neighbourhood** of A if N is an embedded submanifold with boundary and there is a map $r : \partial N \to A$ such that N is homeomorphic to the mapping cylinder M_r under $A \sqcup \partial N \times \{1\}$. To get an idea of what this looks like consider taking $A \subseteq X$ to be either the boundary inclusion $S^1 \subseteq D^2$, or the inclusion of an embedded $S^1 \subseteq S^2$.

[3] This means check continuity too! Sorry.

Now it's easy to see that if A has a mapping cylinder neighbourhood in X, then the inclusion $A \hookrightarrow X$ is a cofibration. Indeed, the mapping cylinder M_r carries an obvious Strøm structure for which φ takes the value 1 on $\partial N \times \{1\}$, and the deformation H_t fixes the points of $\partial N \times \{1\}$ at all times t. This structure is transported to N by the assumed homeomorphism, and can then be extended over all of X in this obvious way.

Of course not every closed subset of X has a mapping cylinder neighbourhood. Neither does every inclusion $A \subseteq X$ which is a cofibration necessarily have a mapping cylinder neighbourhood. What is true is that there is a variety of situtations in which these gadgets can be constructed using differential-topological methods and so recognised.

For instance if X is either a smooth or topological manifold with boundary ∂X, then the inclusion $\partial X \hookrightarrow X$ admits a mapping cylinder neighbourhood, which is then called a **collar**. For a discussion of collars in the topological case see Hatcher [3] Pr. 3.42 for a simple presentation with some compactness assumptions, or Brown [1] for the general case. The smooth case is simpler, and treated in detail by Lee [4] Th. 9.25.

Similarly, if M is a smooth manifold, then the inclusion $N \subseteq M$ of a closed submanifold has a mapping cylinder neighbourhood. In this form it is called a **tubular neighbourhood**, and its existence is discussed in Lee [4] §6. Tubular neighbourhoods also exist under further technical assumptions in the topological case. Details can be found in Brown [1].

Proposition 2.6 *If M is a smooth manifold with boundary, then the inclusion $\partial M \hookrightarrow M$ is a closed cofibration. If M is a smooth manifold and N a closed submanifold, then the inclusion $N \subseteq M$ is a closed cofibration.* ∎

3 Recognising Cofibrations

In this section you will find exercises which display some tricks that can be used to easily spot cofibrations.

Exercise 3.1 Show that if $i : A \hookrightarrow B$ and $j : B \hookrightarrow X$ are cofibrations, then $k = ji : A \hookrightarrow X$ is a cofibration. □

Exercise 3.2 Show that if

$$\begin{tikzcd} A \arrow[r, hook, "j"] \arrow[d, "f"'] & X \arrow[d, "F"] \\ B \arrow[r] & X \cup_A B \end{tikzcd} \tag{3.1}$$

is a pushout square and j is (closed) cofibration, then the map $B \to X \cup_A B$ is a (closed) cofibration. Moreover, show in this case that the induced map $F' : X/A \to (X \cup_A B)/B$ is a homeomorphism. □

Example 3.1 Applications:

- If $j : A \hookrightarrow X$ is a cofibration, then so is $[A] \hookrightarrow X/A$. In particular, the inclusion of the basepoint $* \hookrightarrow S^n \cong D^n/S^{n-1}$ is a cofibration.
- The inclusion $X \hookrightarrow M_j$ is a cofibration, and $M_j/X \cong CA$.

- Combine Exercises 2.2, 3.1 and 3.2 with Example 2.1: If X is a finite dimensional CW complex, then the inclusion $A \hookrightarrow X$ of a subcomplex A is a cofibration. In particular the inclusion of any skeleton $X_n \hookrightarrow X$ is a cofibration. These statements are true in the infinite dimensional case too, but we have to know a little bit of CW topology to make a rigourous induction. $\square$

Exercise 3.3 Let $i : A \hookrightarrow X$ and $j : B \hookrightarrow Y$ be closed cofibrations. Assume that X, Y are compact Hausdorff and show that $i \times j : A \times B \hookrightarrow X \times Y$ is a cofibration. (Hint: First show that if $j : A \to X$ is a cofibration and K is locally compact, then $j \times 1 : A \times K \to X \times K$ is a cofibration. For this you may study (2.2)). $\square$

4 Applications

In this last section we find some applications for the theory we have developed. To put the first in context you may recall the extension problem (2.3).

Exercise 4.1 Let $j : A \hookrightarrow X$ be a cofibration which is the inclusion of a nonempty subspace and let $q : X \to X/A$ be the projection onto the quotient space. Assume that the map j is null homotopic and show that q has a left homotopy inverse. In particular this says that X is a retract of X/A in the homotopy category. Now assume that A is contractible and show that in this case q is a homotopy equivalence. $\square$

Example 4.1 Let X be a pointed space and assume that the inclusion $* \hookrightarrow X$ is a cofibration. We consider both the unreduced suspension $\widetilde{\Sigma}X$ and the reduced suspension ΣX as quotients of $X \times I$. Let $q : X \times I \to \widetilde{\Sigma}X$ be the quotient map and let $\pi : \widetilde{\Sigma}X \to \Sigma X$ be the quotient obtained by collapsing $q(* \times I)$.

Claim: The inclusion $q(* \times I) \hookrightarrow \widetilde{\Sigma}X$ is a cofibration.

Clearly $q(* \times I)$ is contractible, so if the claim is true, then in completing Exercise 4.1 you will have shown that under the assumption on X the quotient map

$$\pi : \widetilde{\Sigma}X \to \Sigma X \tag{4.1}$$

is a homotopy equivalence. To prove the claim we will need a quick lemma.

Lemma 4.1 *The inclusion $i : (* \times I) \cup (X \times \partial I) \hookrightarrow X \times I$ is a cofibration.*

Proof Using Theorem 2.3 we see that the inclusion $k : \partial I \hookrightarrow I$ is a closed cofibration. Choose Strøm structures (φ_X, H_X) for j and (φ_I, H_I) for k. Since k is closed we can assume that $\varphi_I^{-1}(0) = \partial I$. Then the pair

$$\varphi(x, y) = \min\{\varphi_X(x), \varphi_I(t)\}, \qquad H_s(x, t) = (H_X(x, \min\{s, \varphi_I(t)\}), H_I(t, \min\{s, \varphi_X(x)\}))$$

define a Strøm structure for i. $\blacksquare$

Now, the map $\varphi : X \times I \to I$ just produced satisfies $\varphi((* \times I) \cup (X \times \partial I)) = \{0\}$, so factors to give a map $\widetilde{\varphi} : \widetilde{\Sigma} X \to I$ with $\widetilde{\varphi}^{-1}(0) = q(* \times I)$. Define $\widetilde{H} : \Sigma X \times I \to \Sigma X$ by

$$\widetilde{H}_s\langle x, t\rangle = q(H_s(x,t)). \tag{4.2}$$

We can check directly that this is well-defined. To see that it is continuous we can appeal to the fact that $q \times id : (X \times I) \times I \to \widetilde{\Sigma} X \times I$ is a quotient map due to I being locally compact. If $s > \widetilde{\varphi}(\langle x, t\rangle) = \varphi(x,t)$, then

$$\widetilde{H}_s\langle x, t\rangle = q(H_s(x,t)) \in q(* \times I \cup \partial I \times X) = q(* \times I). \tag{4.3}$$

Thus by Strøm's critera we have the original claim. $\square$

Example 4.2 Take the natural numbers and form their one-point compactification $\mathbb{N}_\infty$, this being homeomorphic to the subspace $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \subsetneq \mathbb{R}$. The inclusion $\infty \hookrightarrow \mathbb{N}_\infty$ is *not* a cofibration (cf. Exercise 2.5), and the collapse map $\widetilde{\Sigma}\mathbb{N}_\infty \to \Sigma\mathbb{N}_\infty$ is *not* a homotopy equivalence.

In fact these two spaces are not homotopy equivalent. The unreduced suspension $\widetilde{\Sigma} X$ is an infinite wedge of circles with countable π_1, while ΣX is the so-called Hawaiian earing with uncountable π_1. The Hawaiian earing embeds in $\mathbb{R}^2$ as a shrinking wedge of circles, while $\widetilde{\Sigma} X$ is homotopy equivalent to an exapanding wedge of circles in the plane. What is the difference between $\widetilde{\Sigma}\mathbb{N}_\infty$ and $\Sigma\mathbb{N}_\infty$? The first is $\mathbb{R}_\infty \wedge \mathbb{N}_+$ while the second is $\mathbb{R}_\infty \wedge \mathbb{N}_\infty = (\mathbb{R} \times \mathbb{N})_\infty$. $\square$

Example 4.3 Assume that X is a path-connected CW complex. Then the 0-skeleton X_0 is a discrete set of points and the 1-skeleton X_1 is necessarily a connected graph. This is true since S^{n-1} is path-connected for $n \geq 2$, so if X_1 were not path-connected, then it would not be possible to attach higher dimensional cells to it so as to yield the path-connected X.

Now, with a brief argument we can prove that there is a subcomplex $T \subseteq X_1$ with the properties that *i)* $T_0 = X_0$, and *ii)* T is contractible. The subcomplex T in this context is called a *maximal tree*. The first condition implies that X_1/T_1 is a wedge of circles, one for each edge of X_1 which is not in T. Also we see that $X' = X/T$ is a CW complex whose 0-skeleton is a point. On the other hand, since a subcomplex inclusion is a cofibration, Exercise 4.1 implies that the quotient map

$$X \to X' \tag{4.4}$$

is a homotopy equivalence.

Conclusion: A connected CW complex is homotopy equivalent to a CW complex with a single 0-cell.

We will revisit this example and explain it in more detail in a subsequent lecture. $\square$

The next exercise is particularly important.

Exercise 4.2 Fix a cofibration $j : A \hookrightarrow X$. Let $f, g : A \to B$ be maps and form the pushout squares

$$\begin{tikzcd} A \arrow[r, hook, "j"] \arrow[d, "f"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, hook, "j"] \arrow[d, "g"'] & X \arrow[d] \\ B \arrow[r] & Z. \end{tikzcd} \tag{4.5}$$

Assume given a homotopy $F : f \simeq g : A \to B$. Show that there is a homotopy equivalence $Y \simeq Z$ under B[4] (Hint: you need to have done Exercise 3.2). $\square$

Example 4.4 Take a map $\varphi : S^{n-1} \to A$ and attach a cell to get $X = A \cup_\varphi e^n$. Assume that $\varphi \simeq *$. Then X is homotopy equivalent to $A \cup_* e^n$ under A. In particular

$$X \simeq A \vee S^n. \qquad \square \tag{4.6}$$

Finally, the last exercise of this section serves to clarify some comments made during the first lecture.

Exercise 4.3 *The relevant background for this exercise is found in Example 1.4 of the first lecture.* Show that if $j : A \hookrightarrow X$ is both a cofibration and a weak deformation retract, then it is a strong deformation retract. Let C denote the comb space and use the first part of the exercise to prove the following.

- The inclusion $\{(0,1)\} \hookrightarrow C$ is not a cofibration.
- The inclusion $\{0\} \times I \subseteq C$ is not a cofibration.
- The inclusion $C \subseteq I^2$ is not a cofibration. $\square$

References

[1] M. Brown, *Locally Flat Imbeddings of Topological Manifolds*, Ann. Math., **75**, (1962), 331-341.

[2] T. tom Dieck, K. Kamps, D. Puppe, *Homotopietheorie*, Springer-Verlag Berlin Heidelberg, (1970).

[3] A. Hatcher, *Algebraic Topology*, Cambridge University Press, (2002). Available at http://pi.math.cornell.edu/ hatcher/AT/ATpage.html.

[4] J. Lee, *Introduction to Smooth Manifolds*, Second Edition, Springer, (2003).

[5] A. Strøm, *Note on Cofibrations II*, Math. Scand. **22**, (1968), 130-142.

[4] i.e. $H_t(b) = b$, $\forall b \in B$, $t \in I$.